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**PREPARATION SEQUENCING SITUATIONS AND RELATED  
GAMES**

By Edwin Lohmann, Peter Borm, Marco Slikker

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# Preparation sequencing situations and related games

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## Abstract

In this paper preparation sequencing situations are introduced. This new type of one-machine sequencing situations assumes that before a job can start, some preparation depending on its predecessor is required. Preparation sequencing situations are first analyzed from an operations research perspective: optimality conditions are provided and an algorithm is provided to obtain an optimal order. Secondly, we analyze the allocation problem of the minimal joint cost from a game theoretic perspective. A corresponding preparation sequencing game is defined and the focus is on the core and nucleolus of such games.

keywords: preparation times, sequencing situations, cooperative game theory, core, nucleolus

JEL classification code: C71

## 1 Introduction

Sequencing theory deals with a variety of problems sharing several characteristics: a number of jobs have to be processed on one or more machines, in such a way that a cost criterion is minimized. From one sequencing problem to another the way these characteristics are defined can differ and additional constraints can be added: the machines can be parallel or serial, there can be conditions on the order in which the jobs should be processed and different cost criteria can be used. Applications of the theory of sequencing situations are numerous and diverse: from manufacturing and maintenance to scheduling patients in an operating room.

The starting point of the game theoretic analysis of sequencing situations is the paper by Curiel, Pederzoli, and Tijs (1989). In their one-machine model, only one job can be processed at a time. The processing time is deterministic for every job, and every job has a certain constant cost per time unit it spends in the system. A job is in the system from the moment the machine starts processing the first job until the job itself is processed by the machine. An order that minimizes total cost, processes the jobs in a decreasing order with respect to their urgency (cost per time unit divided by the length of the job, cf. Smith (1956)). A procedure

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is introduced that, given an initial order, uses neighbor switches to obtain the optimal order and constructs a stable cost allocation in the process.

Since Curiel, Pederzoli, and Tijs (1989) several related classes of sequencing problems are discussed, including ready times, due dates, multiple machines and numerous cost criteria (see e.g., Curiel et al. (2002), Borm et al. (2002), Calleja et al. (2002) and Slikker (2005)). The model we will discuss here deals with the phenomenon that the state in which the machine is left behind by a player's predecessor influences the amount of time the player needs before he can start on his actual job. One can think of cleaning up a machine, adjusting a machine to the new jobs or something simple as erasing the blackboard before one can start the lecture. This is captured in our model of preparation sequencing situations: a one-machine sequencing situation, where the time a job spends in the system depends on its predecessor. Moreover, contrary to the basic model, it is assumed that a job just enters the system as soon as the previous job is finished. However, after every job some preparation is required. The amount of time this takes depends on the type of job that was processed before. This way, the cost incurred by a job depends on the duration of the preparation, its own processing time and the cost per time unit. As the costs incurred during the processing of the job itself is constant across all possible orders of jobs, we will not incorporate these costs into our analysis. Verdaasdonk (2007) initiated the study on this subject. In the current paper, we will focus on those preparation sequencing situations where there are only two different values for the preparation time and two different values for the costs per time unit.

The topic will be treated from two perspectives. The first part concerns the operations research perspective. For each preparation sequencing situation we will characterize optimal orders in which joint costs are minimized, and provide an algorithm to obtain an optimal order. The second part, concerning the game theoretic perspective, involves the allocation of the minimal joint costs. We define an objective and consistent way to determine cost savings for each coalition of jobs. For the resulting cooperative game a first focus point is the core. We show that under mild conditions a stable allocation exists. Furthermore, we provide explicit expressions for the core of a large class of preparation sequencing situations. We formulate conditions such that the core of the preparation sequencing game is a singleton. Also, similarities between a special class of assignment games called Böhm-Bawerk horse market games (see e.g., Böhm-Bawerk (1923) and Núñez and Rafels (2005)), and certain preparation sequencing games are pointed out. It is well known that the core of a Böhm-Bawerk horse market game is a line segment (Shapley and Shubik (1971)), where one extreme point is 'buyer'-optimal and the other extreme point is 'seller'-optimal. Under mild conditions, the core of a preparation sequencing situation is a line segment, where we can identify two subsets of the player set, acting as the 'buyers' and the 'sellers'. A second focus point is the nucleolus (Schmeidler (1969), see Peleg and Sudhölter (2003) for a survey on the nucleolus). As the nucleolus is a core-selector, for those preparation sequencing games where the core is a singleton the nucleolus coincides with the core. If the core is a line segment, we provide an explicit expression of the nucleolus in terms of the underlying preparation sequencing situation. As the general expression for the nucleolus heavily depends on the exact parameters, we provide a more basic and less volatile allocation rule called the *large instance based allocation rule* that coincides with the nucleolus for large classes of preparation sequencing games and is also contained in the core for every preparation sequencing game.

Related sequencing problems are discussed by Gupta (1988) and Van der Veen et al. (1998). In Gupta (1988) the mean flow time is minimized in sequencing situations with switching times between jobs depending on the class of both jobs. Not only is the objective

function different from ours, but Gupta (1988) also assumes that the switching time between pair of jobs within the same class is zero. The change-over model by Van der Veen et al. (1998) on the other hand defines the switching time yet in a different way, and minimizes the makespan rather than total costs.

The paper is organized as follows: in the subsequent section, we formally introduce the preparation sequencing model. Also, we provide optimality conditions regarding the processing order, and give an algorithm to find an optimal order. Section 3 contains the game theoretic analysis and focusses on characterizing the core and the nucleolus for preparation sequencing games.

## 2 Preparation sequencing situations

A preparation sequencing situation is defined by a tuple  $\Psi = (N, \alpha, p, p_0)$ . Here,  $N$  denotes the nonempty finite player set. It is assumed that every player owns exactly one job. As there is a one-to-one correspondence between players and jobs, we will use the words player and job interchangeably throughout the paper. The vector  $\alpha \in \mathbb{R}_+^N$  is such that for player  $i \in N$ , the costs of spending  $t$  time units in the system is given by  $\alpha_i t$ . The preparation times are denoted by the vector  $p \in \mathbb{R}_+^N$ , where for  $i \in N$ ,  $p_i$  is the preparation time needed *after* the job of player  $i$  is processed and before the machine can process another job. Note that  $p_i$  denotes the preparation time needed after job  $i \in N$ , and not the processing time of job  $i$ . The time needed before the machine can process the first job is denoted by  $p_0$ .

A processing order on the jobs is described by a bijection  $\sigma : \{1, \dots, |N|\} \rightarrow N$ , where  $\sigma(k)$  denotes the job at position  $k$  in the queue. The set of all orders on  $N$  is denoted by  $\Pi(N)$ . For notational convenience, we set  $\sigma(0) = 0$  and therefore  $p_{\sigma(0)} = p_0$  for all  $\sigma \in \Pi(N)$ .

In preparation sequencing situations, it is assumed that a player enters the system at the moment the machine starts to be prepared for his job and leaves the system as soon as his job is finished. This situation is shown in Figure 1. This differs from standard sequencing problems as depicted in Figure 2, where there are no preparation times and a player enters the system already as the first job in the order starts processing and leaves after his own job is finished.

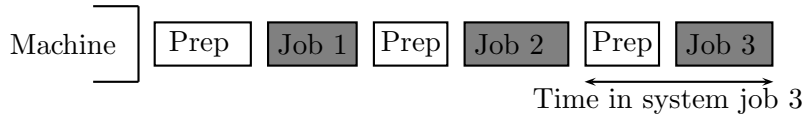


Figure 1: Time in system for preparation sequencing

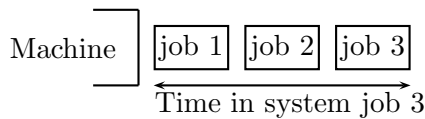


Figure 2: Time in system for standard sequencing

The time a job spends in the system consists of a preparation time depending on the job that is processed before him and his own processing time. The costs arising from this last part is constant over all orders. Hence, we just focus on the costs arising from preparation. So, for an order  $\sigma \in \Pi(N)$  the corresponding costs  $\gamma_i(\sigma)$  for player  $i \in N$  are given by

$$\gamma_i(\sigma) = \alpha_i p_{\sigma(\sigma^{-1}(i)-1)}.$$

	<i>costs</i>	<i>preparation time</i>
$N_h^H$	$\alpha^H$	$p^h$
$N_l^H$	$\alpha^H$	$p^l$
$N_h^L$	$\alpha^L$	$p^h$
$N_l^L$	$\alpha^L$	$p^l$

Table 1: Partition of the player set

For a coalition  $S \in 2^N$ , we set  $\gamma_S(\sigma) = \sum_{i \in S} \gamma_i(\sigma)$ . We call an order  $\sigma^* \in \Pi(N)$  optimal for  $N$  if  $\gamma_N(\sigma^*) = \min\{\gamma_N(\sigma) \mid \sigma \in \Pi(N)\}$ .

In this paper we will restrict ourselves to the analysis of preparation sequencing situations with two different values for the preparation times and two different values for the cost per time unit. We denote by  $Prep^{2,2}$  the class of all preparation sequencing situations satisfying this restriction. So, for every  $(N, \alpha, p, p_0) \in Prep^{2,2}$ , there exist  $\alpha^H, \alpha^L \in \mathbb{R}_+$ ,  $\alpha^H > \alpha^L$  such that for all  $i \in N$  it holds that either  $\alpha_i = \alpha^H$  or  $\alpha_i = \alpha^L$ . With respect to the preparation times, we assume there exist  $p^h, p^l \in \mathbb{R}_+$ ,  $p^h > p^l$ , such that for all  $i \in N \cup \{0\}$  it holds that either  $p_i = p^h$  or  $p_i = p^l$ . We partition the set of players according to their characteristics as provided in Table 1, into sets  $N_h^H$ ,  $N_l^H$ ,  $N_h^L$  and  $N_l^L$ . Note that the superscript refers to the cost per time unit, and the subscript refers to the preparation time. Also, throughout the paper uppercase  $H$  and  $L$  will refer to cost per time unit and lowercase  $h$  and  $l$  to preparation time. We denote  $N^H = N_h^H \cup N_l^H$ , and define  $N^L$ ,  $N_h$  and  $N_l$  in a similar way. For a subset  $S \in 2^N$  we use a similar notation:  $S_h^H = S \cap N_h^H$ ,  $S^H = S \cap N^H$ , etc.

**Example 2.1.** Consider the preparation sequencing problem  $\Psi = (N, \alpha, p, p_0)$ , where  $N = \{1, 2, 3\}$ ,  $\alpha = (\alpha_1, \alpha_2, \alpha_3) = (4, 1, 1)$ ,  $p = (p_1, p_2, p_3) = (3, 3, 1)$  and  $p_0 = 3$ . It is readily checked that  $N_h^H = \{1\}$ ,  $N_l^H = \emptyset$ ,  $N_h^L = \{2\}$  and  $N_l^L = \{3\}$ . The order  $\sigma \in \Pi(N)$  such that  $\sigma(1) = 3$ ,  $\sigma(2) = 2$  and  $\sigma(3) = 1$  gives  $\gamma_1(\sigma) = p_2\alpha_1 = 12$ ,  $\gamma_2(\sigma) = p_3\alpha_2 = 1$ , and  $\gamma_3(\sigma) = p_0\alpha_3 = 3$ , so  $\gamma_N(\sigma) = 16$ . However, the order  $\sigma' \in \Pi(N)$  such that  $\sigma'(1) = 2$ ,  $\sigma'(2) = 3$  and  $\sigma'(3) = 1$  gives  $\gamma_N(\sigma') = 10$ .



Figure 3: The orders  $\sigma$  and  $\sigma'$  for the preparation sequencing situation of Example 2.1.

Naturally, an interesting question is how we can identify whether an order is optimal or not. Also, if we can find sufficient conditions for this, could we use these conditions to construct an optimal order? As it turns out, we can indeed find such conditions and use these to obtain an algorithm that constructs an optimal order for every preparation sequencing situation in  $Prep^{2,2}$ .

First we focus on the sufficient conditions. For this, we introduce the following additional notation. Given a preparation sequencing situation  $(N, \alpha, p, p_0) \in Prep^{2,2}$  and an order  $\sigma \in \Pi(N)$ , define the following classes of neighboring pairs:

$$M^{hH}(\sigma) = \{(i, j) \in (N \cup \{0\}) \times N \mid p_i = p^h, \alpha_j = \alpha^H, \sigma^{-1}(i) = \sigma^{-1}(j) - 1\},$$

$$M^{lL}(\sigma) = \{(i, j) \in (N \cup \{0\}) \times N \mid p_i = p^l, \alpha_j = \alpha^L, \sigma^{-1}(i) = \sigma^{-1}(j) - 1\},$$

$$M^{hL}(\sigma) = \{(i, j) \in (N \cup \{0\}) \times N \mid p_i = p^h, \alpha_j = \alpha^L, \sigma^{-1}(i) = \sigma^{-1}(j) - 1\},$$

and

$$M^{lH}(\sigma) = \{(i, j) \in (N \cup \{0\}) \times N \mid p_i = p^l, \alpha_j = \alpha^H, \sigma^{-1}(i) = \sigma^{-1}(j) - 1\}.$$

Note that the first superscript indicates the preparation time and the second superscript indicates the cost level. For every  $\sigma \in \Pi(N)$ , we have the following equalities:

$$|M^{hH}(\sigma)| + |M^{lH}(\sigma)| = |N^H|, \quad (1)$$

and

$$|M^{lL}(\sigma)| + |M^{hL}(\sigma)| = |N^L|. \quad (2)$$

If  $p_0 = p^h$ , we have for every order  $\sigma \in \Pi(N)$  that

$$|M^{lH}(\sigma)| + |M^{lL}(\sigma)| = |N_l| - \mathbb{1}_{[p_{\sigma(|N|)}=p^l]}, \quad (3)$$

and

$$|M^{hH}(\sigma)| + |M^{hL}(\sigma)| = |N_h| + \mathbb{1}_{[p_{\sigma(|N|)}=p^l]}, \quad (4)$$

since  $\mathbb{1}_{[p_{\sigma(|N|)}=p^l]} = 1 - \mathbb{1}_{[p_{\sigma(|N|)}=p^h]}$ ,  $p_0 = p^h$  and the preparation time of the last player in the order does not incur costs for a player in  $N$ . If  $p_0 = p^l$ , we have

$$|M^{lH}(\sigma)| + |M^{lL}(\sigma)| = |N_l| + \mathbb{1}_{[p_{\sigma(|N|)}=p^h]}, \quad (5)$$

and

$$|M^{hH}(\sigma)| + |M^{hL}(\sigma)| = |N_h| - \mathbb{1}_{[p_{\sigma(|N|)}=p^h]}, \quad (6)$$

for every order  $\sigma \in \Pi(N)$ . Again, note that  $\mathbb{1}_{[p_{\sigma(|N|)}=p^h]} = 1 - \mathbb{1}_{[p_{\sigma(|N|)}=p^l]}$ . The following theorem states the sufficient conditions for an order to be optimal.

**Theorem 2.2.** *Let  $(N, \alpha, p, p_0) \in \text{Prep}^{2,2}$  and let  $\sigma \in \Pi(N)$ . If  $p_{\sigma(|N|)} = \max_{i \in N} p_i$  and either  $|M^{hH}(\sigma)| = 0$  or  $|M^{lL}(\sigma)| = 0$ , then  $\sigma$  is optimal.*

*Proof.* Assume  $p_0 = p^h$  and first consider the case where  $\max_{i \in N} p_i = p^l$ . This implies that  $|N_h^H| = |N_h^L| = 0$ . For  $\sigma \in \Pi(N)$  we obtain

$$\gamma_N(\sigma) = \sum_{i \in N} p^l \alpha_i + (p_0 - p^l) \alpha_{\sigma(1)} = \sum_{i \in N} p^l \alpha_i + (p^h - p^l) \alpha_{\sigma(1)}.$$

Take an order  $\sigma' \in \Pi(N)$  such that  $|M^{hH}(\sigma')| = 0$ . As  $p_0 = p^h$ , we obtain that  $\alpha_{\sigma'(1)} = \alpha^L$  and therefore  $\sigma'(1) \in N_l^L$ . Hence,  $\gamma_N(\sigma') = \sum_{i \in N} p^l \alpha_i + (p^h - p^l) \alpha^L \leq \gamma_N(\sigma)$  for all  $\sigma \in \Pi(N)$ , so  $\sigma'$  is optimal.

Now take an order  $\sigma'' \in \Pi(N)$  such that  $|M^{lL}(\sigma'')| = 0$  and  $|M^{hH}(\sigma'')| > 0$ . Then either  $|N_l^L| = 0$ , which means that  $N = N_l^H$  and every order is optimal, or  $|N_l^L| = 1$  with  $\sigma''(1) \in N_l^L$ . In the last case  $\gamma_N(\sigma'') = \sum_{i \in N} p^l \alpha_i + (p_0 - p^l) \alpha^L \leq \gamma_N(\sigma)$  for all  $\sigma \in \Pi(N)$ , and  $\sigma''$  is optimal.

Now consider the case where  $\max_{i \in N} p_i = p^h$ . Take an arbitrary  $\sigma \in \Pi(N)$ . Take  $B, D \in \mathbb{N}$  such that  $B = |M^{hH}(\sigma)|$  and  $D = |M^{lL}(\sigma)|$ . Note that by Equation (2) and (4) we have

$$B - D = |M^{hH}(\sigma)| - |M^{lL}(\sigma)| = |N_h^H| - |N_l^L| + \mathbb{1}_{[p_{\sigma(|N|)}=p^l]}.$$

By (3) and (4) it holds that

$$\begin{aligned} \gamma_N(\sigma) &= |M^{hH}(\sigma)|p^h\alpha^H + |M^{lH}(\sigma)|p^l\alpha^H + |M^{hL}(\sigma)|p^h\alpha^L + |M^{lL}(\sigma)|p^l\alpha^L \\ &= Bp^h\alpha^H + (|N_l| - \mathbb{1}_{[p_{\sigma(|N|)}=p^l]} - D)p^l\alpha^H \\ &\quad + (|N_h| + \mathbb{1}_{[p_{\sigma(|N|)}=p^l]} - B)p^h\alpha^L + Dp^l\alpha^L \\ &\geq (B - \min\{B, D\})p^h\alpha^H + (|N_l| - \mathbb{1}_{[p_{\sigma(|N|)}=p^l]} - D + \min\{B, D\})p^l\alpha^H \\ &\quad + (|N_h| + \mathbb{1}_{[p_{\sigma(|N|)}=p^l]} - B + \min\{B, D\})p^h\alpha^L + (D - \min\{B, D\})p^l\alpha^L \\ &= \max\{0, |N_h^H| - |N_l^L| + \mathbb{1}_{[p_{\sigma(|N|)}=p^l]}\}p^h\alpha^H \\ &\quad + \min\{|N_l| - \mathbb{1}_{[p_{\sigma(|N|)}=p^l]}, |N^H|\}p^l\alpha^H \\ &\quad + \min\{|N^L|, |N_h| + \mathbb{1}_{[p_{\sigma(|N|)}=p^l]}\}p^h\alpha^L \\ &\quad + \max\{|N_l^L| - |N_h^H| - \mathbb{1}_{[p_{\sigma(|N|)}=p^l]}, 0\}p^l\alpha^L \\ &\geq \max\{0, |N_h^H| - |N_l^L|\}p^h\alpha^H + \min\{|N_l|, |N^H|\}p^l\alpha^H \\ &\quad + \min\{|N^L|, |N_h| + \mathbb{1}_{[p_{\sigma(|N|)}=p^l]}\}p^h\alpha^L + \max\{|N_l^L| - |N_h^H|, 0\}p^l\alpha^L \end{aligned}$$

where the inequalities follow from the observation that  $(p^h - p^l)(\alpha^H - \alpha^L) > 0$ . The first inequality holds with equality if either  $B = 0$  or  $D = 0$ , and the second inequality holds with equality if  $p_{\sigma(|N|)} = p^h$ . This shows that an order  $\sigma \in \Pi(N)$  with  $p_{\sigma(|N|)} = p^h$  and either  $|M^{hH}(\sigma)| = 0$  or  $|M^{lL}(\sigma)| = 0$  is optimal.

The case where  $p_0 = p^l$  can be proven analogously, using Equation (5) and (6) instead of Equation (3) and (4).  $\square$

The optimality conditions in Theorem 2.2 consist of two parts: the first condition states that it is optimal to place a player with highest preparation time possible at the last position. The second condition means that it is optimal to place players with low costs behind players with high preparation time and players with high costs behind players with low preparation time. These conditions are used in the following algorithm. The first condition is explicitly taken care of in step 2, the second condition is dealt with in step 3. Step 4 deals with these optimality conditions more implicitly, which is demonstrated in Example 2.3.

### Algorithm 1

*Input:* a preparation sequencing situation  $(N, \alpha, p, p_0) \in \text{Prep}^{2,2}$ .

*Output:* an order  $\tilde{\sigma} \in \Pi(N)$ .

*Step 1.* Initialize  $s = 1$  and  $C_1^1 = N$

*Step 2.* Define

$$C_s^2 = \begin{cases} C_s^1 \setminus N_h & \text{if } |C_s^1 \cap N_h| = 1 \text{ and } s \neq |N|, \\ C_s^1 & \text{else.} \end{cases}$$



Step 3. Define

$$C_s^3 = \begin{cases} C_s^2 \cap N^L & \text{if } p_{\tilde{\sigma}(s-1)} = p^h \text{ and } C_s^2 \cap N^L \neq \emptyset, \\ C_s^2 \cap N^H & \text{if } p_{\tilde{\sigma}(s-1)} = p^l \text{ and } C_s^2 \cap N^H \neq \emptyset, \\ C_s^2 & \text{else.} \end{cases}$$

Step 4. Define

$$C_s^4 = \begin{cases} C_s^3 \cap N_l & \text{if } C_s^3 \cap N_l \neq \emptyset, \\ C_s^3 & \text{else.} \end{cases}$$

Step 5. Choose a job  $i \in C_s^4$  and define  $\tilde{\sigma}(s) = i$ .

Step 6. If  $s = |N|$ , stop.

If  $s < |N|$ , set  $s = s + 1$  and, subsequently, set  $C_s^1 = C_{s-1}^1 \setminus \{\tilde{\sigma}(s-1)\}$ . Next, return to step 2.

The notation  $\tilde{\sigma}$  is used for an order provided by the algorithm. The algorithm generates this order by filling up all positions in the order from front to back. For every position, the set of candidate players is narrowed down in a few steps. In the algorithm,  $C_s^4 \subseteq C_s^3 \subseteq C_s^2 \subseteq C_s^1 \in 2^N$  are the sets of candidate players for the  $s^{th}$  position in this order.

**Example 2.3.** Consider the preparation sequencing situation  $\Psi = (N, \alpha, p, p_0)$ , where we have  $\alpha = (2, 2, 1, 1)$ ,  $p = (3, 1, 3, 1)$  and  $p_0 = 3$ . We have  $N_h^H = \{1\}$ ,  $N_l^H = \{2\}$ ,  $N_h^L = \{3\}$ , and  $N_l^L = \{4\}$ . As  $|N_h| = 2$ , we have  $C_1^2 = C_1^1 = N$  (see Table 2). In step 3 we obtain  $C_1^3 = \{3, 4\}$  as  $p_0 = p^h$ . Step 4 further narrows down the set of candidate players for the first position, as  $C_1^4 = \{4\}$ . Therefore, we obtain  $\tilde{\sigma}(1) = 4$ . Now that player 4 is placed, we have  $C_2^2 = C_2^1 = \{1, 2, 3\}$ . In step 3 of iteration 2, we obtain  $C_2^3 = \{1, 2\}$  and in step 4 we obtain  $C_2^4 = \{2\}$  so  $\tilde{\sigma}(2) = 2$ . In the third iteration,  $C_3^2 = C_3^1 = \{1, 3\}$  and  $C_3^4 = C_3^3 = \{1\}$  so  $\tilde{\sigma}(3) = 1$  and  $\tilde{\sigma}(4) = 3$ .

$s$	1	2	3	4
$C_s^1$	$N$	$\{1, 2, 3\}$	$\{1, 3\}$	$\{3\}$
$C_s^2$	$N$	$\{1, 2, 3\}$	$\{1, 3\}$	$\{3\}$
$C_s^3$	$\{3, 4\}$	$\{1, 2\}$	$\{1\}$	$\{3\}$
$C_s^4$	$\{4\}$	$\{2\}$	$\{1\}$	$\{3\}$
$\tilde{\sigma}(s)$	4	2	1	3

Table 2: Sets of candidate players in Example 2.3

It is easily seen that a player with high preparation time is placed last. Furthermore, players with high costs are placed behind players with low preparation time and the other way around (see Figure 4). We obtain  $\gamma_N(\tilde{\sigma}) = 10$  which is indeed optimal. Also, note the importance of step 4 of the algorithm: if we would place an arbitrary player in  $C_2^3$  at position 2, we could have ended up with the order  $\sigma'$  such that  $\sigma'(1) = 4$ ,  $\sigma'(2) = 1$ ,  $\sigma'(3) = 2$ ,  $\sigma'(4) = 3$ , with  $\gamma_N(\sigma') = 12$ .

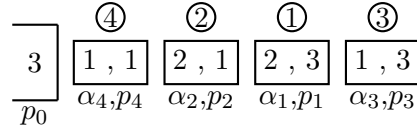


Figure 4: Order  $\tilde{\sigma}$  provided by Algorithm 1 in Example 2.3

Now we are ready to prove that Algorithm 1 provides an optimal order, for every preparation sequencing situation  $(N, \alpha, p, p_0) \in \text{Prep}^{2,2}$ .

**Theorem 2.4.** *Let  $\Psi = (N, \alpha, p, p_0) \in \text{Prep}^{2,2}$ . Then Algorithm 1 provides an optimal order for  $N$ .*

*Proof.* Let  $\tilde{\sigma} \in \Pi(N)$  be an order provided by Algorithm 1. In Step 2 of the algorithm, it is made sure that there is always a player with the highest available preparation time left to place at the last position. Hence,  $p_{\tilde{\sigma}(|N|)} = p^h$ , unless  $N_h^H \cup N_h^L = \emptyset$  which implies that there is in fact only one value for  $p_i$  and  $p_{\tilde{\sigma}(|N|)} = p^l = \max_{i \in N} p_i$ .

We will prove that either  $|M^{hH}(\tilde{\sigma})| = 0$  or  $|M^{lL}(\tilde{\sigma})| = 0$ , since this would via Theorem 2.2 imply that  $\tilde{\sigma}$  is optimal.

Assume on the contrary that both  $|M^{hH}(\tilde{\sigma})| > 0$  and  $|M^{lL}(\tilde{\sigma})| > 0$ . Then there exist  $s, r \in \{0, \dots, |N| - 1\}$  such that  $(\tilde{\sigma}(s), \tilde{\sigma}(s+1)) \in M^{hH}(\tilde{\sigma})$  and  $(\tilde{\sigma}(r), \tilde{\sigma}(r+1)) \in M^{lL}(\tilde{\sigma})$ .

Assume  $r < s$ . According to the algorithm, job  $\tilde{\sigma}(r+1)$  is only placed behind job  $\tilde{\sigma}(r)$  if there is no job  $j$  with  $\alpha_j = \alpha^H$  left that is not yet placed, or there is only one job  $j$  with  $\alpha_j = \alpha^H$  left, but this job has to be reserved for the last spot because it is the only remaining job with high preparation. In the first case, we have a contradiction, since job  $\tilde{\sigma}(s+1)$  is not yet placed. The second case also results in a contradiction, since both  $p_{\tilde{\sigma}(s)} = p^h$  and  $p_{\tilde{\sigma}(|N|)} = p^h$ .

Now assume  $s < r$ . According to the algorithm, job  $\tilde{\sigma}(s+1)$  is only placed behind job  $\tilde{\sigma}(s)$  if there is no job  $j$  with  $\alpha_j = \alpha^L$  left that is not yet placed, or there is only one job  $j$  with  $\alpha_j = \alpha^L$  left, but this job has to be reserved for the last spot because it is the only remaining job with high preparation. In the first case, we have a contradiction, since job  $\tilde{\sigma}(r+1)$  is not yet placed.

The second case can only hold if  $r+1 = |N|$ . For all jobs  $i \in \{\tilde{\sigma}(s+1), \dots, \tilde{\sigma}(r)\}$  it then must hold that  $p_i = p^l$ , otherwise job  $\tilde{\sigma}(r+1)$  would have been placed at position  $s+1$ . Furthermore,  $\alpha_i = \alpha^H$  otherwise job  $i$  would have been placed at position  $s+1$  as this would avoid the combination of  $p^h$  and  $\alpha^H$ . So, we obtain that  $i \in N_l^H$  for all  $i \in \{\tilde{\sigma}(s+1), \dots, \tilde{\sigma}(r)\}$ . Since the algorithm first places the jobs in  $N_l^H$  before placing the jobs in  $N_h^H$ , and  $\tilde{\sigma}(|N|) \notin N_h^H$ , we obtain that  $N_h^H = \emptyset$ . Furthermore, if there existed a job  $i \in N_l^L$  then the algorithm would place every job in  $N_l^H$  directly behind this job. But since  $\tilde{\sigma}(s) \notin N_l^L$  this implies that  $N_l^L = \emptyset$ . Hence, the second case only allows players in  $N_l^H$  and  $N_h^L$ , so  $N_h^H \cup N_l^L = \emptyset$ . The solution provided by the algorithm for this situation (first all players in  $N_h^L$  but one, then all players in  $N_l^H$  and finally the last player in  $N_h^L$ ) is clearly optimal.  $\square$

The proof of Theorem 2.4 implies the following.

**Remark 2.5.** *If  $N_h^H \cup N_l^L \neq \emptyset$ , then every order provided by the algorithm satisfies the sufficient conditions of Theorem 2.2.*

### 3 Preparation sequencing games

In the previous section we addressed the problem of finding an optimal order for preparation sequencing situations. An additional question is how the total costs of such an optimal order should be allocated among the players. To answer this question, we will use the framework of transferable utility games. Let us first recall some basic concepts within cooperative game theory used in the later analysis.

A transferable utility game  $(N, v)$  is defined by a finite player set  $N$  and a function  $v$  on the set  $2^N$  of all subsets of  $N$  assigning to each coalition  $S \in 2^N$  a value  $v(S) \in \mathbb{R}$  such that  $v(\emptyset) = 0$ . The imputation set  $I(v)$  of a game  $(N, v)$  is given by all individually rational and efficient allocations, so

$$I(v) = \{x \in \mathbb{R}^N \mid \sum_{i \in N} x_i = v(N), x_i \geq v(\{i\}) \text{ for all } i \in N\}.$$

For a game  $(N, v)$ , the core  $C(v)$  is defined as the set of those imputations, for which no coalition has an incentive to split off:

$$C(v) = \{x \in \mathbb{R}^N \mid \sum_{i \in N} x_i = v(N), \sum_{i \in S} x_i \geq v(S) \text{ for all } S \in 2^N\}.$$

A game  $(N, v)$  is called balanced if its core is nonempty. A game  $(N, v)$  is called convex if  $v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T)$  for all  $i \in N$  and  $S \subset T \subset N \setminus \{i\}$ . Every convex game has a nonempty core. If for  $i \in N$  and  $j \in N$  it holds that  $v(S \cup \{i\}) = v(S \cup \{j\})$  for all  $S \in 2^{N \setminus \{i, j\}}$ , then player  $i$  and  $j$  are called symmetric.

We define the excess of coalition  $S \in 2^N$  with respect to allocation  $x \in I(v)$  by  $E(S, x) = v(S) - \sum_{i \in S} x_i$ . The excess measures the dissatisfaction of coalition  $S$  with respect to allocation  $x$ . Let  $\omega(x) \in \mathbb{R}^{2^{|N|}}$  be the vector of excesses of  $x \in I(v)$ , arranged in weakly decreasing order. For a game  $(N, v)$  such that  $I(v) \neq \emptyset$ , the nucleolus  $\eta(v)$  (Schmeidler (1969)) is the unique imputation  $x \in I(v)$  such that  $\omega(x)$  is lexicographically smaller than  $\omega(y)$  for all  $y \in I(v)$ . So, the nucleolus is the individual rational and efficient allocation that minimizes the highest dissatisfactions in a hierarchical manner. For every game  $(N, v)$  such that  $C(v) \neq \emptyset$ , we have  $\eta(v) \in C(v)$ . Furthermore, if player  $i \in N$  and player  $j \in N$  are symmetric in the game  $(N, v)$  then  $\eta_i(v) = \eta_j(v)$ .

We will limit the game theoretic analysis of preparation sequencing situations to those instances of  $Prep^{2,2}$  where  $p_0 = p^h$ , denoted by  $Prep_h^{2,2}$ . We assume that by cooperating, every coalition  $S \in 2^N \setminus \{\emptyset\}$  can form any order  $\sigma \in \Pi(S)$ . Thus we employ a pessimistic view, in the sense that the preparation time for the first player in the order  $\sigma$  equals  $p_0 = p^h$ . This setup allows us to measure the value of every coalition consistently over all coalitions, and independent of the players outside the coalition. This approach cannot be used for the case where  $p_0 = p^l$  as this would result in an optimistic assessment of the cost of a coalition.

Let  $\Psi = (N, \alpha, p, p_0)$  be a preparation sequencing situation. We will define the costs for coalition  $S \in 2^N \setminus \{\emptyset\}$  as the costs of an optimal order in the preparation sequencing problem  $(S, \alpha', p', p'_0)$ , where  $\alpha' \in \mathbb{R}^S$  and  $p' \in \mathbb{R}^{S \cup \{0\}}$  are such that  $\alpha'_i = \alpha_i$  for all  $i \in S$ , and  $p'_i = p_i$  for all  $i \in S \cup \{0\}$ . Given a preparation sequencing situation  $\Psi = (N, \alpha, p, p_0)$  and a coalition  $S \in 2^N \setminus \{\emptyset\}$ , we denote by  $\tilde{\sigma}_S$  an optimal order of the situation  $(S, \alpha', p', p'_0)$ . Hence, formally,

the preparation sequencing game  $(N, v^\Psi)$  is defined by

$$v^\Psi(S) = \sum_{i \in S} \gamma_i(\tilde{\sigma}_{\{i\}}) - \gamma_S(\tilde{\sigma}_S),$$

for all  $S \in 2^N \setminus \{\emptyset\}$ . Clearly,  $v^\Psi(\{i\}) = 0$  for every  $i \in N$ .

**Example 3.1.** *Reconsider the preparation sequencing situation of Example 2.3, where  $\alpha = (2, 2, 1, 1)$ ,  $p = (3, 1, 3, 1)$  and  $p_0 = 3$ . It is easily seen that  $\gamma_1(\tilde{\sigma}_{\{1\}}) = \gamma_2(\tilde{\sigma}_{\{2\}}) = 6$  and  $\gamma_3(\tilde{\sigma}_{\{3\}}) = \gamma_4(\tilde{\sigma}_{\{4\}}) = 3$ . Take  $S = \{1, 2, 4\}$ . The optimal order  $\tilde{\sigma}_S$  is such that  $\tilde{\sigma}_S(1) = 4$ ,  $\tilde{\sigma}_S(2) = 2$ , and  $\tilde{\sigma}_S(3) = 1$ , which results in total costs  $\gamma_S(\tilde{\sigma}_S) = 7$ . Hence, we have  $v^\Psi(S) = (6 + 6 + 3) - 7 = 8$ . Table 3 displays the value of every coalition. Note that  $(N, v^\Psi)$  is not convex, since  $v^\Psi(\{3, 4\}) - v^\Psi(\{4\}) = 2 > 0 = v^\Psi(N) - v^\Psi(\{1, 2, 4\})$ .*

$S$	1	2	3	4	12	13	14	23	24	34	123	124	134	234	$N$
$v(S)$	0	0	0	0	4	0	4	2	2	2	4	8	4	6	8

Table 3: The game  $(N, v^\Psi)$  of Example 3.1

For  $\Psi = (N, \alpha, p, p_0) \in \text{Prep}_h^{2,2}$  we can explicitly express the value of each coalition in terms of the number of players in the different player classes in the preparation sequencing situation.

**Lemma 3.2.** *Let  $\Psi = (N, \alpha, p, p_0) \in \text{Prep}_h^{2,2}$ . Then*

$$v^\Psi(S) = \begin{cases} (|S_l^H| + |S_l^L|)(p^h - p^l)\alpha^H & \text{if } S_h^H \neq \emptyset \text{ and } |S_h^H| \geq |S_l^L|; \\ |S_h^H|(p^h - p^l)(\alpha^H - \alpha^L) + |S_l^H|(p^h - p^l)\alpha^H & \text{if } S_h^H \neq \emptyset \text{ and } |S_h^H| < |S_l^L|; \\ + |S_l^L|(p^h - p^l)\alpha^L & \\ |S_l^H|(p^h - p^l)\alpha^H + |S_l^L|(p^h - p^l)\alpha^L & \text{if } S_h^H = \emptyset, S_l^L \neq \emptyset \text{ and } S_h^L \neq \emptyset; \\ |S_l^H|(p^h - p^l)\alpha^H + (|S_l^L| - 1)(p^h - p^l)\alpha^L & \text{if } S_h^H = \emptyset, S_l^L \neq \emptyset \text{ and } S_h^L = \emptyset; \\ (|S_l^H| - 1)(p^h - p^l)\alpha^H + (p^h - p^l)\alpha^L & \text{if } S_h^H = \emptyset, S_l^L = \emptyset, S_l^H \neq \emptyset \\ & \text{and } S_h^L \neq \emptyset; \\ (|S_l^H| - 1)(p^h - p^l)\alpha^H & \text{if } S_h^H = \emptyset, S_l^L = \emptyset \text{ and } S_h^L = \emptyset; \\ 0 & \text{if } S_h^H = \emptyset, S_l^L = \emptyset \text{ and } S_l^H = \emptyset \end{cases}$$

for all  $S \in 2^N \setminus \{\emptyset\}$ .

*Proof.* We will only prove the first case, the expressions for the other cases follow from a similar reasoning. Take  $S \in 2^N$  such that  $S_h^H \neq \emptyset$  and  $|S_h^H| \geq |S_l^L|$ . First of all,  $\sum_{i \in S} \gamma_i(\tilde{\sigma}_{\{i\}}) = (|S_h^H| + |S_l^H|)p^h\alpha^H + (|S_h^L| + |S_l^L|)p^h\alpha^L$ . Since  $S_h^H \neq \emptyset$ , we know that for every optimal order  $\tilde{\sigma}_S$  it holds that  $p_{\tilde{\sigma}_S(|S|)} = p^h$ . Furthermore, since  $S_h^H \neq \emptyset$  we have by Remark 2.5 that either  $|M^{hH}(\tilde{\sigma}_S)| = 0$  or  $|M^{lL}(\tilde{\sigma}_S)| = 0$ . By equations (1) – (4), it must hold that  $|M^{lL}(\tilde{\sigma}_S)| = 0$ , since  $|M^{hH}(\tilde{\sigma}_S)| = 0$  would imply that  $|M^{lL}(\tilde{\sigma}_S)| < 0$ . So, we have

$$\begin{aligned} \gamma_S(\tilde{\sigma}_S) &= |M^{hH}(\tilde{\sigma}_S)|p^h\alpha^H + |M^{lH}(\tilde{\sigma}_S)|p^l\alpha^H + |M^{hL}(\tilde{\sigma}_S)|p^h\alpha^L + |M^{lL}(\tilde{\sigma}_S)|p^l\alpha^L, \\ &= (|S_h^H| - |S_l^L|)p^h\alpha^H + (|S_l^H| + |S_l^L|)p^l\alpha^H + (|S_h^L| + |S_l^L|)p^h\alpha^L, \end{aligned}$$

and we may conclude that

$$\begin{aligned}
v^\Psi(S) &= \sum_{i \in S} \gamma_i(\tilde{\sigma}_{\{i\}}) - \gamma_S(\tilde{\sigma}_S), \\
&= (|S_h^H| + |S_l^H|)p^h\alpha^H + (|S_h^L| + |S_l^L|)p^h\alpha^L, \\
&\quad - [(|S_h^H| - |S_l^L|)p^h\alpha^H + (|S_l^H| + |S_l^L|)p^l\alpha^H + (|S_h^L| + |S_l^L|)p^h\alpha^L], \\
&= (|S_l^L| + |S_l^H|)(p^h - p^l)\alpha^H.
\end{aligned}$$

□

All marginal contributions can be readily determined from Lemma 3.2.

**Corollary 3.3.** *Let  $\Psi = (N, \alpha, p, p_0) \in \text{Prep}_h^{2,2}$ ,  $i \in N$  and  $S \in 2^{N \setminus \{i\}}$ . If  $i \in N_h^H$ ,*

$$v^\Psi(S \cup \{i\}) - v^\Psi(S) = \begin{cases} (p^h - p^l)\alpha^H & \text{if } S \neq \emptyset, S_h^H = \emptyset \text{ and } S_h^L = \emptyset; \\ (p^h - p^l)(\alpha^H - \alpha^L) & \text{if } S_h^H \neq \emptyset \text{ and } |S_h^H| < |S_l^L|, \\ & \text{or if } S_h^H = \emptyset, S_l^L \neq \emptyset \text{ and } S_h^L \neq \emptyset, \\ & \text{or if } S_h^H = \emptyset, S_l^L = \emptyset, S_l^H \neq \emptyset \text{ and } S_h^L \neq \emptyset; \\ 0 & \text{otherwise.} \end{cases}$$

If  $i \in N_l^L$ ,

$$v^\Psi(S \cup \{i\}) - v^\Psi(S) = \begin{cases} 0 & \text{if } S = \emptyset; \\ (p^h - p^l)\alpha^H & \text{if } S_h^H \neq \emptyset \text{ and } |S_h^H| > |S_l^L|, \\ & \text{or if } S_h^H = \emptyset, S_l^L = \emptyset \text{ and } S_l^H \neq \emptyset; \\ (p^h - p^l)\alpha^L & \text{otherwise.} \end{cases}$$

If  $i \in N_l^H$ ,

$$v^\Psi(S \cup \{i\}) - v^\Psi(S) = \begin{cases} 0 & \text{if } S = \emptyset; \\ (p^h - p^l)\alpha^L & \text{if } S_h^H = \emptyset, S_l^L = \emptyset, S_l^H = \emptyset \text{ and } S_h^L \neq \emptyset; \\ (p^h - p^l)\alpha^H & \text{otherwise.} \end{cases}$$

Finally, if  $i \in N_h^L$ ,

$$v^\Psi(S \cup \{i\}) - v^\Psi(S) = \begin{cases} (p^h - p^l)\alpha^L & \text{if } S \neq \emptyset, S_h^H = \emptyset \text{ and } S_h^L = \emptyset; \\ 0 & \text{otherwise.} \end{cases}$$

The following example illustrates both Lemma 3.2 and Corollary 3.3.

**Example 3.4.** *Consider the preparation sequencing situation  $\Psi = (N, \alpha, p, p_0) \in \text{Prep}_h^{2,2}$  such that  $N = \{1, 2, 3, 4, 5, 6\}$ ,  $\alpha = (5, 5, 5, 2, 2, 2)$ ,  $p = (3, 3, 1, 3, 1, 1)$  and  $p_0 = 3$ . This means that  $N_h^H = \{1, 2\}$ ,  $N_l^H = \{3\}$ ,  $N_h^L = \{4\}$  and  $N_l^L = \{5, 6\}$ . Consider the coalition  $S = \{1, 3, 6\}$ . The stand-alone costs for these players are  $\gamma_1(\tilde{\sigma}_{\{1\}}) = \gamma_3(\tilde{\sigma}_{\{3\}}) = 15$  and  $\gamma_6(\tilde{\sigma}_{\{6\}}) = 6$ . Using*

Algorithm 1, we obtain  $\tilde{\sigma}_S(1) = 6$ ,  $\tilde{\sigma}_S(2) = 3$  and  $\tilde{\sigma}_S(3) = 2$  for the optimal order  $\tilde{\sigma}_S$  for coalition  $S$ . Hence,  $\gamma_S(\tilde{\sigma}_S) = 16$  and  $v^\Psi(S) = \sum_{i \in S} \gamma_i(\tilde{\sigma}_{\{i\}}) - \gamma_S(\tilde{\sigma}_S) = 20$ . This is in line with Lemma 3.2, as  $(|S_l^H| + |S_l^L|)(p^h \alpha^H - p^l \alpha^H) = 2(15 - 5) = 20$ . By Corollary 3.3, we have  $v(\{2, 3, 6\}) - v(\{3, 6\}) = (p^h - p^l) \alpha^H = 10$  which is easily verified, as  $v(\{3, 6\}) = 10$ . On the other hand, the marginal contribution of player 2 when he joins coalition  $\{1, 3, 6\}$  equals zero, as  $v(\{1, 2, 3, 6\}) = v(\{1, 3, 6\})$ .

We will use the expressions from Lemma 3.2 to show that every preparation sequencing game has a nonempty core. To this end, we define the *large instance based allocation rule*  $\theta$  on the class of preparation sequencing situations  $Prep_h^{2,2}$ .

**Definition 3.5.** Let  $\Psi = (N, \alpha, p, p_0) \in Prep_h^{2,2}$ . Then, for all  $i \in N$ ,

$$\theta_i(\Psi) = (p^h - p_i) \alpha_i + \begin{cases} \frac{(\alpha^H - \alpha^L)(p^h - p^l)}{2} & \text{if } i \in N_h^H \cup N_l^L \text{ and } |N_h^H| = |N_l^L|; \\ (\alpha^H - \alpha^L)(p^h - p^l) & \text{if } i \in N_h^H \text{ and } |N_h^H| < |N_l^L|, \\ & \text{or if } i \in N_l^L \text{ and } |N_h^H| > |N_l^L|; \\ -\frac{1}{|N_l^L|}(p^h - p^l) \alpha^L & \text{if } i \in N_l^L \text{ and } |N_h^H| = 0; \\ -\frac{1}{|N_l^H|}(\alpha^H - \alpha^L)(p^h - p^l) & \text{if } i \in N_l^H, |N_h^L| > 0 \text{ and} \\ & |N_h^H| = |N_l^L| = 0; \\ -\frac{1}{|N_l^H|}(p^h - p^l) \alpha^H & \text{if } i \in N_l^H \text{ and } N = N_l^H; \\ 0 & \text{otherwise.} \end{cases}$$

The common part of the expression for  $\theta(\Psi)$ ,  $(p^h - p_i) \alpha_i$ , gives an estimation of the cost savings that can be attributed to player  $i$ . This estimation is based on the marginal costs of player  $i$  entering in a fictive, ‘large’ coalition. The part  $p^h \alpha_i$  are the stand-alone costs of player  $i$ . Now assume there is an order  $\sigma \in \Pi(N)$  where  $p_{\sigma(k)} = p_i$  for some  $k$ . If player  $i$  is placed in between player  $\sigma(k)$  and player  $\sigma(k+1)$ , then the marginal costs equal  $p_i \alpha_i$ . Hence, we estimate the cost savings by  $(p^h - p_i) \alpha_i$ .

The second part serves as a correction to this estimation: a player in  $N_h^H$  and a player in  $N_l^L$  together are responsible for more cost savings than we already allocated to them. These additional cost savings go to the minority, the players in  $N_h^H$  if  $|N_h^H| < |N_l^L|$  and the players in  $N_l^L$  if  $|N_l^L| < |N_h^H|$ , and is shared equally if  $|N_h^H| = |N_l^L|$ . The other corrections are due to ‘small’ instances: for example, if there are no players in both  $N_h^H$  and  $N_l^L$ , then the cost savings attributed to players in  $N_l^H$  is overestimated, and is corrected.

**Example 3.6.** Reconsider the preparation sequencing situation  $\Psi$  of Example 3.4. We have  $|N_h^H| = |N_l^L| > 0$ , so  $\theta(\Psi) = (3, 3, 10, 0, 7, 7)$ .

For every preparation sequencing situation  $\Psi \in Prep_h^{2,2}$ , the large instance based allocation rule provides a core element for the corresponding game  $(N, v^\Psi)$ .

**Theorem 3.7.** Let  $\Psi = (N, \alpha, p, p_0) \in Prep_h^{2,2}$ . Then  $\theta(\Psi) \in C(v^\Psi)$ .

*Proof.* We consider four different cases, and use Lemma 3.2 and Definition 3.5 in each of these cases.

(i) Assume  $|N_h^H| = 0$ ,  $|N_l^L| = 0$  and  $|N_h^L| > 0$ . For  $S \in 2^N \setminus \{\emptyset\}$  we have

$$\begin{aligned}
\theta_S(\Psi) - v^\Psi(S) &\geq \sum_{i \in S_l^H} \left( (p^h - p^l)\alpha^H - \frac{1}{|N_l^H|} \cdot (p^h - p^l)(\alpha^H - \alpha^L) \right) \\
&\quad - (|S_l^H| - 1)(p^h - p^l)\alpha^H - (p^h - p^l)\alpha^L \\
&= |S_l^H| \left( (p^h - p^l)\alpha^H - \frac{1}{|N_l^H|} \cdot (p^h - p^l)(\alpha^H - \alpha^L) \right) \\
&\quad - (|S_l^H| - 1)(p^h - p^l)\alpha^H - (p^h - p^l)\alpha^L \\
&= (1 - \frac{|S_l^H|}{|N_l^H|})(p^h - p^l)(\alpha^H - \alpha^L) \\
&\geq 0,
\end{aligned}$$

with equality if  $S = N$ , and therefore  $\theta(\Psi) \in C(v^\Psi)$ .

(ii) Assume  $|N_h^H| + |N_l^L| > 0$  and  $|N_h| > 0$ . For  $S \in 2^N \setminus \{\emptyset\}$  we have

$$\begin{aligned}
\theta_S(\Psi) - v^\Psi(S) &\geq |S_l^H|(p^h - p^l)\alpha^H + |S_l^L|(p^h - p^l)\alpha^L \\
&\quad + \min\{|S_h^H|, |S_l^L|\}(p^h - p^l)(\alpha^H - \alpha^L) \\
&\quad - |S_l^H|(p^h - p^l)\alpha^H + |S_l^L|(p^h - p^l)\alpha^L \\
&\quad - \min\{|S_h^H|, |S_l^L|\}(p^h - p^l)(\alpha^H - \alpha^L) \\
&\geq 0,
\end{aligned}$$

again with equality if  $S = N$ , and therefore  $\theta(\Psi) \in C(v^\Psi)$ .

(iii) Assume  $|N_h| = 0$  and  $|N_l^L| > 0$ . For  $S \in 2^N \setminus \{\emptyset\}$  we have

$$\begin{aligned}
\theta_S(\Psi) - v^\Psi(S) &\geq |S_l^H|(p^h - p^l)\alpha^H + |S_l^L|((p^h - p^l)\alpha^L - \frac{1}{|N_l^L|} \cdot (p^h - p^l)\alpha^L) \\
&\quad - |S_l^H|(p^h - p^l)\alpha^H - (|S_l^L| - 1)(p^h - p^l)\alpha^L \\
&= (1 - \frac{|S_l^L|}{|N_l^L|})(p^h - p^l)\alpha^L \\
&\geq 0,
\end{aligned}$$

with equality if  $S = N$ , and therefore  $\theta(\Psi) \in C(v^\Psi)$ .

(iv) Finally, assume  $|N_h^H| = |N_l^L| = |N_h^L| = 0$ . For  $S \in 2^N \setminus \{\emptyset\}$  we have

$$\begin{aligned}
\theta_S(\Psi) - v^\Psi(S) &= |S_l^H|((p^h - p^l)\alpha^H - \frac{1}{|N_l^H|} \cdot (p^h - p^l)\alpha^H) \\
&\quad - (|S_l^H| - 1)(p^h - p^l)\alpha^H \\
&= (1 - \frac{|S_l^H|}{|N_l^H|})(p^h - p^l)\alpha^H \\
&\geq 0,
\end{aligned}$$

with equality if  $S = N$ , and therefore  $\theta(\Psi) \in C(v^\Psi)$ . □

For every preparation sequencing situation  $\Psi = (N, \alpha, p, p_0) \in \text{Prep}_h^{2,2}$  such that  $|N_h^H| = |N_l^L| > 0$ , the core is the convex hull of two vectors. We will show this in Theorem 3.8. This structure of the core is similar to the one for a specific type of assignment games, called Böhm-Bawerk horse market games (Böhm-Bawerk 1923), in the sense that the core consist of a line segment, where one extreme point is ‘buyer’-optimal and the other extreme point is ‘seller’-optimal. In our setting, the players in  $N_h^H$  act as the buyers and players in  $N_l^L$  act as the sellers. Translated into the terminology of horse market games, every player in  $N_h^H$  is interested to buy the right on low preparation, and every player in  $N_l^L$  is interested to sell the right on low preparation. By interacting on the market, every pair of a buyer and a seller creates a profit that can be shared in an arbitrary, nonnegative way. This profit equals  $(p^h - p^l)(\alpha^H - \alpha^L)$ .

For a preparation sequencing situation  $\Psi = (N, \alpha, p, p_0) \in \text{Prep}_h^{2,2}$  and  $j \in N$ , define

$$\begin{aligned}\bar{\theta}_j(\Psi) &= (p^h - p_j)\alpha_j + \begin{cases} (p^h - p^l)(\alpha^H - \alpha^L) & \text{if } j \in N_h^H; \\ 0 & \text{else,} \end{cases} \\ \underline{\theta}_j(\Psi) &= (p^h - p_j)\alpha_j + \begin{cases} (p^h - p^l)(\alpha^H - \alpha^L) & \text{if } j \in N_l^L; \\ 0 & \text{else.} \end{cases}\end{aligned}$$

Note that both  $\bar{\theta}(\Psi)$  and  $\underline{\theta}(\Psi)$  are efficient and

$$\theta(\Psi) = \frac{1}{2}(\bar{\theta}(\Psi) + \underline{\theta}(\Psi)).$$

Note that  $\bar{\theta}(\Psi)$  corresponds to the ‘buyer’-optimal allocation, and  $\underline{\theta}(\Psi)$  to the ‘seller’-optimal allocation.

**Theorem 3.8.** *Let  $\Psi = (N, \alpha, p, p_0) \in \text{Prep}_h^{2,2}$  be such that  $|N_h^H| = |N_l^L| > 0$ . Then  $C(v^\Psi) = \text{Conv}\{\bar{\theta}(\Psi), \underline{\theta}(\Psi)\}$ .*

*Proof.* First we establish that  $\text{Conv}\{\bar{\theta}(\Psi), \underline{\theta}(\Psi)\} \subseteq C(v^\Psi)$ , by showing that  $\bar{\theta}(\Psi) \in C(v^\Psi)$  and  $\underline{\theta}(\Psi) \in C(v^\Psi)$ .

Let  $S \in 2^N \setminus \{\emptyset\}$ . Then we have that

$$\begin{aligned}\sum_{i \in S} \bar{\theta}_i(\Psi) - v^\Psi(S) &= \sum_{i \in S} (p^h - p_i)\alpha_i + |S_h^H|(p^h - p^l)(\alpha^H - \alpha^L) - v^\Psi(S) \\ &\geq |S_l^H|(p^h - p^l)\alpha^H + |S_l^L|(p^h - p^l)\alpha^L + |S_h^H|(p^h - p^l)(\alpha^H - \alpha^L) \\ &\quad - \left( |S_l^H| \cdot (p^h - p^l)\alpha^H + |S_l^L|(p^h - p^l)\alpha^L \right) \\ &\quad + \min\{|S_h^H|, |S_l^L|\}(p^h - p^l)(\alpha^H - \alpha^L) \\ &= (|S_h^H| - \min\{|S_h^H|, |S_l^L|\})(p^h - p^l)(\alpha^H - \alpha^L) \\ &\geq 0.\end{aligned}$$

Note that the first inequality follows from Lemma 3.2, and that for  $S = N$  the two inequalities hold with equality. Hence,  $\bar{\theta}(\Psi) \in C(v^\Psi)$ . A similar reasoning shows that  $\underline{\theta}(\Psi) \in C(v^\Psi)$ .

Now we show that  $C(v^\Psi) \subseteq \text{Conv}\{\bar{\theta}(\Psi), \underline{\theta}(\Psi)\}$ . It suffices to show that for every  $x \in C(v^\Psi)$  it holds that:

- (i)  $x_i \geq 0$  for every  $i \in N$ .



- (ii)  $x_i = 0$  for every  $i \in N_h^L$ .
- (iii)  $x_i = (p^h - p^l)\alpha^H$  for every  $i \in N_l^H$ .
- (iv)  $x_i + x_j = (p^h - p^l)\alpha^H$  for every  $i \in N_h^H$  and  $j \in N_l^L$  and, therefore,  
 $x_i = x_k$  for all  $i, k \in N_h^H$ , and  $x_j = x_r$  for all  $j, r \in N_l^L$ .
- (v)  $(p^h - p^l)\alpha^L \leq x_i \leq (p^h - p^l)\alpha^H$  for every  $i \in N_l^L$ .

Note that (iv) and (v) together imply  $0 \leq x_i \leq (p^h - p^l)(\alpha^H - \alpha^L)$  for every  $i \in N_h^H$ . We will prove (i) - (v) point by point. Take  $x \in C(v^\Psi)$ .

- (i) As  $x_i \geq v(\{i\})$  and  $v(\{i\}) = 0$  for every  $i \in N$ , we have  $x_i \geq 0$  for every  $i \in N$ .
- (ii) As  $v^\Psi(N \setminus N_h^L) = v^\Psi(N)$ , we obtain  $x_i = 0$  for all  $i \in N_h^L$ .
- (iii) Take  $i \in N_l^H$ . Then

$$\begin{aligned}
\sum_{j \in N} x_j + x_i &= \sum_{j \in N_h^H \cup \{i\}} x_j + \sum_{j \in N \setminus N_h^H} x_j \\
&\geq v^\Psi(N_h^H \cup \{i\}) + v^\Psi(N \setminus N_h^H) \\
&= (|N_l^L| + 1)(p^h - p^l)\alpha^H + |N_l^H|(p^h - p^l)\alpha^H \\
&= v^\Psi(N) + (p^h - p^l)\alpha^H \\
&= \sum_{j \in N} x_j + (p^h - p^l)\alpha^H.
\end{aligned}$$

Hence,  $x_i \geq (p^h - p^l)\alpha^H$ . By Corollary 3.3, we have  $v^\Psi(N) - v^\Psi(N \setminus \{i\}) = (p^h - p^l)\alpha^H$  and therefore  $x_i = (p^h - p^l)\alpha^H$ .

- (iv) Take  $i \in N_h^H$  and  $j \in N_l^L$ . As  $v^\Psi(\{i, j\}) = (p^h - p^l)\alpha^H = v^\Psi(N) - v^\Psi(N \setminus \{i, j\})$  we have  $x_i + x_j = (p^h - p^l)\alpha^H$ .
- (v) By (iv), we have that  $x_i = x_j$  for all  $i, j \in N_l^L$ . Moreover, since

$$v^\Psi(N_h^L \cup N_l^L) + v^\Psi(N_h^H \cup N_l^H \cup N_l^L) = |N_l^L|(p^h - p^l)\alpha^L + v^\Psi(N),$$

it follows that  $x_i \geq (p^h - p^l)\alpha^L$  for every  $i \in N_l^L$ . Furthermore, since

$$v(N) - v(N \setminus N_l^L) = |N_l^L|(p^h - p^l)\alpha^H,$$

it follows that  $x_i \leq (p^h - p^l)\alpha^H$ .

□

Not only for preparation sequencing situations with  $|N_h^H| = |N_l^L| > 0$  we can find an easy expression for the structure of the core of the corresponding game. If the player set contains at least one player of every type and  $|N_h^H| \neq |N_l^L|$ , the large instance based allocation rule turns out to be the only core element.

**Theorem 3.9.** Let  $\Psi = (N, \alpha, p, p_0) \in \text{Prep}_h^{2,2}$  be such that  $N_h^H, N_l^H, N_h^L$  and  $N_l^L$  are all nonempty, and  $|N_h^H| \neq |N_l^L|$ . Then  $\theta(\Psi)$  is the only core element of  $(N, v^\Psi)$  and, consequently,  $\theta(\Psi) = \eta(v^\Psi)$ .

*Proof.* For every  $x \in C(v^\Psi)$  and  $i \in N$  it holds that  $x_i \leq v^\Psi(N) - v^\Psi(N \setminus \{i\})$ . By efficiency of  $\theta$ , it suffices to show that  $\theta_i(\Psi) = v^\Psi(N) - v^\Psi(N \setminus \{i\})$  for every  $i \in N$ .

First consider the case  $|N_h^H| > |N_l^L|$ . By Corollary 3.3 and Definition 3.5 we obtain that

$$v^\Psi(N) - v^\Psi(N \setminus \{i\}) = \begin{cases} 0 & \text{if } i \in N_h^H; \\ (p^h - p^l)\alpha^H & \text{if } i \in N_l^L; \\ (p^h - p^l)\alpha^H & \text{if } i \in N_l^H; \\ 0 & \text{if } i \in N_h^L; \end{cases} \\ = \theta_i(\Psi).$$

Now consider the case  $|N_l^L| > |N_h^H|$ . Then, by Corollary 3.3 and Definition 3.5,

$$v^\Psi(N) - v^\Psi(N \setminus \{i\}) = \begin{cases} (p^h - p^l)(\alpha^H - \alpha^L) & \text{if } i \in N_h^H; \\ (p^h - p^l)\alpha^L & \text{if } i \in N_l^L; \\ (p^h - p^l)\alpha^H & \text{if } i \in N_l^H; \\ 0 & \text{if } i \in N_h^L; \end{cases} \\ = \theta_i(\Psi).$$

□

The coincidence between the large instance based allocation rule  $\theta$  and the nucleolus can be extended to preparation sequencing situations as considered in Theorem 3.8 with one further restriction.

**Theorem 3.10.** Let  $\Psi = (N, \alpha, p, p_0) \in \text{Prep}_h^{2,2}$  be a preparation sequencing situation such that  $|N_h^H| = |N_l^L| > 1$ . Then  $\theta(\Psi) = \eta(v^\Psi)$ .

*Proof.* We show that  $\theta(\Psi) = \eta(v^\Psi)$  by showing that the vector of excesses of  $\theta(\Psi)$  is lexicographically smaller than the vector of excesses of any other core element. By Theorem 3.8,  $C(v^\Psi) = \text{Conv}\{\bar{\theta}(\Psi), \underline{\theta}(\Psi)\}$ . So, take  $c \in [0, 1]$  and define  $x^c = c\underline{\theta}(\Psi) + (1 - c)\bar{\theta}(\Psi)$ . By Lemma 3.2, the excesses are

$$E(S, x^c) = \begin{cases} -c(|S_h^H| - |S_l^L|)A & \text{if } S_h^H \neq \emptyset \text{ and } |S_h^H| \geq |S_l^L|; \\ -(1 - c)(|S_l^L| - |S_h^H|)A & \text{if } S_h^H \neq \emptyset \text{ and } |S_h^H| < |S_l^L|; \\ -(1 - c)|S_l^L|A & \text{if } S_h^H = \emptyset, S_l^L \neq \emptyset \text{ and } S_h^L \neq \emptyset; \\ -(1 - c)|S_l^L|A - (p^h - p^l)\alpha^L & \text{if } S_h^H = \emptyset, S_l^L \neq \emptyset \text{ and } S_h^L = \emptyset; \\ -A & \text{if } S_h^H = \emptyset, S_l^L = \emptyset, S_l^H \neq \emptyset \text{ and } S_h^L \neq \emptyset; \\ -(p^h - p^l)\alpha^H & \text{if } S_h^H = \emptyset, S_l^L = \emptyset, S_l^H \neq \emptyset \text{ and } S_h^L = \emptyset; \\ 0 & \text{if } S_h^H = \emptyset, S_l^L = \emptyset \text{ and } S_l^H = \emptyset; \end{cases}$$

where  $A = (p^h - p^l)(\alpha^H - \alpha^L) > 0$ .

It is readily checked that the highest excess equals 0. This excess occurs, independent of the value of  $c$ , for every coalition  $S \in 2^N \setminus \{\emptyset\}$  such that either  $|S_h^H| = |S_l^L| > 0$  or  $S = S_h^L$ . For  $c = 0$  or  $c = 1$ , there are additional coalitions with excess equal to zero whereas for  $c \in (0, 1)$  all other coalitions have a negative excess. Hence, both  $x^0 \neq \eta(v^\Psi)$  and  $x^1 \neq \eta(v^\Psi)$ . Since  $-(p^h - p^l)\alpha^H < 0$  and  $-(p^h - p^l)\alpha^L < 0$ , for  $c \in (0, 1)$  the second highest excess equals either  $-cA$  or  $-(1 - c)A$ , or a multiple of these values. Hence, the second highest excess is minimized for  $c = \frac{1}{2}$ , implying that  $\eta(v) = x^{\frac{1}{2}} = \theta(\Psi)$ .  $\square$

In general, the large instance based allocation rule does not coincide with the nucleolus. In fact, the expression for the nucleolus becomes quite involved as it depends not only on the number of players of every type as in the definition of  $\theta(N, \alpha, p, p_0)$ , but on the specific values for  $p^h, p^l, \alpha^H$ , and  $\alpha^L$  as well.

**Example 3.11.** Consider a preparation sequencing situation  $\Psi = (N, \alpha, p, p_0)$  such that  $|N_h^H| = 0$ ,  $|N_l^L| = 0$ ,  $|N_l^H| > 1$  and  $|N_h^L| = 1$ . By Lemma 3.2 we have

$$v^\Psi(N) = (|N_l^H| - 1)(p^h - p^l)\alpha^H + (p^h - p^l)\alpha^L.$$

By the observation that all players in  $N_l^H$  are symmetric, we have that the nucleolus is of the form

$$x_i^\mu(v^\Psi) = \begin{cases} \frac{1}{|N_l^H|}(v^\Psi(N) - \mu) & \text{if } i \in N_l^H; \\ \mu & \text{if } i \in N_h^L, \end{cases}$$

for some  $\mu \in \mathbb{R}$ . Let  $j$  be the unique element of  $N_h^L$ . The excesses are

$$E(S, x^\mu) = \begin{cases} 0 & \text{if } S = \emptyset; \\ \frac{|S_l^H|}{|N_l^H|}((p^h - p^l)(\alpha^H - \alpha^L) + \mu) - (p^h - p^l)\alpha^H & \text{if } S_l^H \neq \emptyset \text{ and } j \notin S; \\ \frac{|S_l^H| - |N_l^H|}{|N_l^H|}((p^h - p^l)(\alpha^H - \alpha^L) + \mu) & \text{if } S_l^H \neq \emptyset \text{ and } j \in S; \\ -\mu & \text{if } S_l^H = \emptyset \text{ and } j \in S; \end{cases}$$

Using a similar approach as in the proof of Theorem 3.10, we obtain that the relevant excesses are given by  $-\mu$ ,  $-(p^h - p^l)\alpha^L + \mu$  and  $-\frac{1}{|N_l^H|}((p^h - p^l)(\alpha^H - \alpha^L) + \mu)$ . For  $|N| \leq \frac{2\alpha^H}{\alpha^L}$ , the maximum of these excesses is minimized by taking  $\mu = \frac{1}{2}(p^h - p^l)\alpha^L$ . If  $|N| > \frac{2\alpha^H}{\alpha^L}$ ,  $\mu = \frac{|N_l^H|}{|N_l^H| + 1}(p^h - p^l)\alpha^H$  minimizes the maximum of these excesses. Hence, we have

$$\eta_i(v^\Psi) = \begin{cases} \frac{1}{2}(p^h - p^l)\alpha^L & \text{if } i = j \text{ and } |N| \leq \frac{2\alpha^H}{\alpha^L}; \\ (p^h - p^l)(\alpha^L - \frac{\alpha^H}{|N_l^H| + 1}) & \text{if } i \in N_l^H \text{ and } |N| > \frac{2\alpha^H}{\alpha^L}; \\ (p^h - p^l)(\alpha^H \frac{|N_l^H| - 1}{|N_l^H|} + \frac{\alpha^L}{2|N_l^H|}) & \text{if } i = j \text{ and } |N| \leq \frac{2\alpha^H}{\alpha^L}; \\ \frac{|N_l^H|}{|N_l^H| + 1}(p^h - p^l)\alpha^H & \text{if } i \in N_h^L \text{ and } |N| > \frac{2\alpha^H}{\alpha^L}. \end{cases}$$

It is readily checked that  $\theta(\Psi) = x^0$ , i.e. it distributes  $v^\Psi(N)$ , irrespective of the specific values of the parameters  $\alpha^H$  and  $\alpha^L$ , equally over the players in  $N_l^H$ , while the unique player in  $N_h^L$  obtains 0.

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